UNITED STATES
NAVAL POSTGRADUATE SCHOOL

A DETAILED INTEGRATION FOR THE RADIATION IMPEDANCE OF A RHOMBIC ANTENNA

---BY---

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ASSISTANT PROFESSOR OF ELECTRONICS

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ABSTRACT

The generalized circuit equation derived by Chaney [1] is applied directly to evaluation of the radiation impedance of a terminated rhombic antenna in free space. The fact that certain integrations totalize to zero, as demonstrated by Chaney [2] from his equations of constraint, is verified by actual detailed evaluation of the integrals involved. Furthermore, Chaney's formulae [2] for the radiation impedances of both the terminated rhombic and vee in free space are verified by independent integration.

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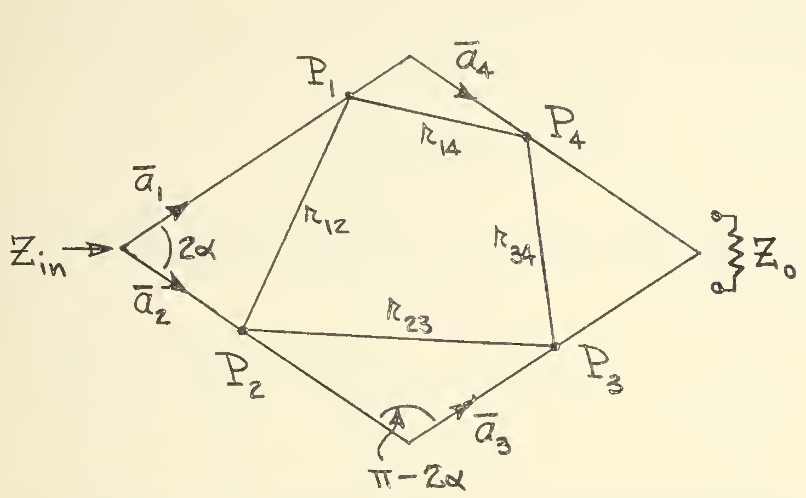
1. J. G. Chaney, "A critical study of the circuit concept", J. Appl. Phys. 22, 12, 1429 (1951).

2. J. G. Chaney, "Free space radiation impedance of rhombic antenna", U. S. Naval Postgraduate School Technical Report No. 4 (BuShips Project Report No. 1)(May 1952).

RADIATION IMPEDANCE OF THE TERMINATED RHOMBIC IN FREE SPACE



$$\left.\begin{aligned}\bar{a}_1 &= \bar{a}_3 \\ \bar{a}_2 &= \bar{a}_4\end{aligned}\right\} \quad \begin{aligned}\bar{a}_1\cdot\bar{a}_1 &= \bar{a}_2\cdot\bar{a}_2 = 1 \\ \bar{a}_1\cdot\bar{a}_2 &= \cos 2\alpha\end{aligned}$$

$$\bar{I}_1 = \bar{a}_1 I_0 e^{-jkx_1} \qquad \bar{I}_3 = -\bar{a}_1 I_0 e^{-jk(l+x_3)}$$

$$\bar{I}_2 = -\bar{a}_2 I_0 e^{-jkx_2} \qquad \bar{I}_4 = \bar{a}_2 \bar{I}_0 e^{-jk(l+x_4)}$$

where l = length of any leg.

Chaney has derived the following formula for the input impedance of a closed circuit[(1)],

$$Z_{in} = Z_0|f_0|^2 + 4lZ_i f_M^2 + \underbrace{j\frac{30}{k}\oint_a\oint_b \mathrm{Re}[f(P_a)^* f(P_b)]\,\Diamond_a[e(r_{ab})\,d\bar{r}_b]\cdot d\bar{r}_a}_{\stackrel{D}{=}\, Z_r \,=\, \text{radiation impedance}}$$

Thus,

$$Z_r = j\frac{30}{k}\sum_{i,j=1}^{4}\oint_i\oint_j \mathrm{Re}[f(P_i)^* f(P_j)]\,\Diamond_i[e(r_{ij})\,d\bar{r}_j]\cdot d\bar{r}_i$$

where $\Diamond \stackrel{D}{=} \nabla(\nabla\cdot\ ) + k^2$, $e(r_{ij}) \stackrel{D}{=} \dfrac{e^{-jkr_{ij}}}{r_{ij}}$, $f(P_i) \stackrel{D}{=} I_i$.

$$\Diamond_i\left[e(r_{ij})\,d\bar{r}_j\right] = \nabla_i\underbrace{\left[\nabla_i\cdot e(r_{ij})\,d\bar{r}_j\right]}_{} + k^2 e(r_{ij})\,d\bar{r}_j$$

$$= \nabla_i e(r_{ij})\cdot d\bar{r}_j + e(r_{ij})\underbrace{\nabla_i\cdot d\bar{r}_j}_{\equiv 0}$$

$$= -\nabla_j e(r_{ij})\cdot d\bar{r}_j$$

$$= -\frac{\partial}{\partial x_j}e(r_{ij})\,dx_j$$

$$Z_R = \frac{30}{jk}\sum_{i,j=1}^{4}\oint_i\oint_j \mathrm{Re}\left[f(P_i)^* f(P_j)\right]\left(\frac{\partial^2}{\partial x_i\partial x_j} - k^2\bar{a}_i\cdot\bar{a}_j\right)e(r_{ij})\,dx_i\,dx_j$$

$$= \frac{30}{jk}\left\{4\oint_1\oint_1 \cos k(x_1-x_1')\left(\frac{\partial^2}{\partial x_1\partial x_1'} - k^2\right)e(r_{11})\,dx_1\,dx_1'\right.$$

$$-2\oint_1\oint_2 \cos k(x_1-x_2)\left(\frac{\partial^2}{\partial x_1\partial x_2} - k^2\cos 2\alpha\right)e(r_{12})\,dx_1\,dx_2$$

$$-2\oint_3\oint_4 \cos k(x_3-x_4)\left(\frac{\partial^2}{\partial x_3\partial x_4} - k^2\cos 2\alpha\right)e(r_{34})\,dx_3\,dx_4$$

$$-2\oint_1\oint_3 \cos k(x_1-x_3-l)\left(\frac{\partial^2}{\partial x_1\partial x_3} - k^2\right)e(r_{13})\,dx_1\,dx_3$$

$$-2\oint_2\oint_4 \cos k(x_2-x_4-l)\left(\frac{\partial^2}{\partial x_2\partial x_4} - k^2\right)e(r_{24})\,dx_2\,dx_4$$

$$+2\oint_1\oint_4 \cos k(x_1-x_4-l)\left(\frac{\partial^2}{\partial x_1\partial x_4} - k^2\cos 2\alpha\right)e(r_{14})\,dx_1\,dx_4$$

$$\left.+2\oint_2\oint_3 \cos k(x_2-x_3-l)\left(\frac{\partial^2}{\partial x_2\partial x_3} - k^2\cos 2\alpha\right)e(r_{23})\,dx_2\,dx_3\right\}$$

$$\oint_1\oint_2 = \oint_3\oint_4\,,\quad \oint_1\oint_3 = \oint_2\oint_4\,,\quad \oint_1\oint_4 = \oint_2\oint_3$$

$$r_{11} = \sqrt{a^2+(x_1-x_1')^2}$$

$$r_{12} = \sqrt{a^2+x_1^2+x_2^2-2x_1x_2\cos 2\alpha}$$

$$r_{34} = \sqrt{a^2+(l-x_3)^2+(l-x_4)^2-2(l-x_3)(l-x_4)\cos 2\alpha}$$

$$r_{13} = \sqrt{a^2 + (x_3 - x_1 + l\cos 2\alpha)^2 + (l\sin 2\alpha)^2}$$

$$r_{24} = \sqrt{a^2 + (x_4 - x_2 + l\cos 2\alpha)^2 + (l\sin 2\alpha)^2}$$

$$r_{14} = \sqrt{a^2 + (l - x_1)^2 + x_4^2 + 2(l - x_1)x_4\cos 2\alpha}$$

$$r_{23} = \sqrt{a^2 + (l - x_2)^2 + x_3^2 + 2(l - x_2)x_3\cos 2\alpha}$$

Define:

$$Z_{11} = \frac{30}{jk}\int_0^l\int_0^l \cos k(x_1 - x_1')\left(\frac{\partial^2}{\partial x_1 \partial x_1'} - k^2\right)e(r_{11})\,dx_1\,dx_1'$$

$$Z_{12} = \frac{30}{jk}\int_0^l\int_0^l \cos k(x_1 - x_2)\left(\frac{\partial^2}{\partial x_1 \partial x_2} - k^2\cos 2\alpha\right)e(r_{12})\,dx_1\,dx_2$$

$$Z_{13} = \frac{30}{jk}\int_0^l\int_0^l \cos k(x_1 - x_3 - l)\left(\frac{\partial^2}{\partial x_1 \partial x_3} - k^2\right)e(r_{13})\,dx_1\,dx_3$$

$$Z_{14} = \frac{30}{jk}\int_0^l\int_0^l \cos k(x_1 - x_4 - l)\left(\frac{\partial^2}{\partial x_1 \partial x_4} - k^2\cos 2\alpha\right)e(r_{14})\,dx_1\,dx_4$$

Then

$$Z_R = 4(Z_{11} - Z_{12} - Z_{13} + Z_{14})$$

Subst. $\cos 2\alpha = 1 - 2\sin^2\alpha$

$$Z_{12} = \frac{30}{jk}\left[\int_0^l\int_0^l \cos k(x_1 - x_2)\left(\frac{\partial^2}{\partial x_1 \partial x_2} - k^2\right)e(r_{12})\,dx_1\,dx_2 + 2k^2\sin^2\alpha\int_0^l\int_0^l \cos k(x_1 - x_2)\,e(r_{12})\,dx_1\,dx_2\right]$$

Z_{14} = "similar expression"

Let

$$A_{ij} = \int_0^l\int_0^l F_{ij}(x_i, x_j)\left(\frac{\partial^2}{\partial x_i \partial x_j} - k^2\right)e(r_{ij})\,dx_i\,dx_j$$

$$B_{ij} = \int_0^l\int_0^l F_{ij}(x_i, x_j)\,e(r_{ij})\,dx_i\,dx_j$$

Then

$$Z_R = \frac{120}{jk}\left[A_{11} - A_{12} - A_{13} + A_{14} - 2k^2\sin^2\alpha\,B_{12} + 2k^2\sin^2\alpha\,B_{14}\right]$$

$$A_{11} = \int_0^l \int_0^l \cos k(x_1 - x_1')\left(\frac{\partial^2}{\partial x_1 \partial x_1'} - k^2\right) e(r_{11})\, dx_1\, dx_1'$$

where $e(r_{11}) = \dfrac{\epsilon^{-jkr_{11}}}{r_{11}}$, $r_{11} = \sqrt{a^2 + (x_1 - x_1')^2}$

$$A_{11} = \int_0^l \int_0^l \cos k(x_1 - x_1') \frac{\partial^2}{\partial x_1 \partial x_1'} e(r_{11})\, dx_1\, dx_1'$$

$$- k^2 \int_0^l \int_0^l \cos k(x_1 - x_1') e(r_{11})\, dx_1\, dx_1'$$

$$= A_{11}' + A_{11}''$$

Integrating by parts

$$A_{11}' = \int_0^l \left[\cos k(x_1 - x_1') \frac{\partial}{\partial x_1'} e(r_{11})\right]_0^l dx_1' + k \int_0^l \int_0^l \sin k(x_1 - x_1') \frac{\partial}{\partial x_1'} e(r_{11})\, dx_1\, dx_1'$$

$$= \int_0^l \cos k(l - x_1') \frac{\partial}{\partial x_1'} e(r_{l1})\, dx_1' - \int_0^l \cos kx_1' \frac{\partial}{\partial x_1'} e(r_{01})\, dx_1'$$

$$+ k \int_0^l \left[\sin k(x_1 - x_1') e(r_{11})\right]_0^l dx_1 + k^2 \int_0^l \int_0^l \cos k(x_1 - x_1') e(r_{11})\, dx_1\, dx_1'$$

$$= \cos k(l - x_1') e(r_{l1}) \Big|_0^l - k \int_0^l \sin k(l - x_1') e(r_{l1})\, dx_1'$$

$$- \cos kx_1' e(r_{01}) \Big|_0^l - k \int_0^l \sin kx_1' e(r_{01})\, dx_1'$$

$$+ k \int_0^l \sin k(x_1 - l) e(r_{1l})\, dx_1 - k \int_0^l \sin kx_1 e(r_{10})\, dx_1$$

$$+ k^2 \int_0^l \int_0^l \cos k(x_1 - x_1') e(r_{11})\, dx_1\, dx_1'$$

Combining

$$A_{11} = A_{11}' + A_{11}'' = 2e(r_{ll}) - 2\cos kl\, e(r_{l0}) - 2k \int_0^l \sin kx_1 e(r_{10})\, dx_1$$

$$- 2k \underbrace{\int_0^l \sin k(l - x_1) e(r_{1l})\, dx_1}_{= \int_0^l \sin kx_1 e(r_{10})\, dx_1}$$

$$A_{11} = 2e(r_{\ell\ell}) - 2\cos k\ell\, e(r_{\ell 0}) - 4k\int_0^\ell \sin kx_1\, e(r_{10})\,dx_1$$

Evaluate integral in last term

$$\int_0^\ell \sin kx_1\, e(r_{10})\,dx_1 = \int_0^\ell \frac{\sin kx_1 \cos kr_{10}}{r_{10}}\,dx_1 - j\int_0^\ell \frac{\sin kx_1 \sin kr_{10}}{r_{10}}\,dx_1$$

$$= \frac{1}{2}\int_0^\ell \frac{\sin k(r_{10}+x_1)}{r_{10}}\,dx_1 + \frac{1}{2}\int_0^\ell \frac{\sin k(r_{10}-x_1)}{r_{10}}\,dx_1$$

$$+ \frac{j}{2}\int_0^\ell \frac{\cos k(r_{10}+x_1)}{r_{10}}\,dx_1 - \frac{j}{2}\int_0^\ell \frac{\cos k(r_{10}-x_1)}{r_{10}}\,dx_1$$

The following well known integration formulae may be readily verified by the transformation $u = r \pm z$ (+ or − as required).

$$\int_{z_1}^{z_2} \frac{\cos\beta(r+z)}{r}\,dz = Ci\,\beta(r_2+z_2) - Ci\,\beta(r_1+z_1)$$

$$\int_{z_1}^{z_2} \frac{\sin\beta(r+z)}{r}\,dz = Si\,\beta(r_2+z_2) - Si\,\beta(r_1+z_1)$$

$$\int_{z_1}^{z_2} \frac{\cos\beta(r-z)}{r}\,dz = Ci\,\beta(r_1-z_1) - Ci\,\beta(r_2-z_2)$$

$$\int_{z_1}^{z_2} \frac{\sin\beta(r-z)}{r}\,dz = Si\,\beta(r_1-z_1) - Si\,\beta(r_2-z_2)$$

where

$$r = \sqrt{z^2+\rho^2}$$

$$\rho = \text{const.}$$

and

$$Si\,z \overset{D}{=} \int_0^z \frac{\sin y}{y}\,dy$$

$$Ci\,z \overset{D}{=} \int_\infty^z \frac{\cos y}{y}\,dy$$

$$\therefore \int_0^\ell \sin kx_1\, e(r_{10})\,dx_1$$

$$= \frac{1}{2}\Big\{\underbrace{Si\,k(r_{\ell 0}+\ell)}_{=Si\,2k\ell} - \cancel{Si\,ka} - Si\,k(\cancel{r_{\ell 0}}-\ell) + \cancel{Si\,ka}$$

$$+ j\Big[\underbrace{Ci\,k(r_{\ell 0}+\ell)}_{=Ci\,2k\ell} - \underbrace{Ci\,ka}_{=C+\ln ka} + \underbrace{Ci\,k(r_{\ell 0}-\ell)}_{=C+\ln k(\sqrt{a^2+\ell^2}-\ell)} - Ci\,ka\Big]\Big\}$$

$$= C + \ln k\left(\frac{a^2}{2\ell}\right)$$

$$= \frac{1}{2}\Big\{Si\,2k\ell + j\big[Ci\,2k\ell - C - \ln 2k\ell\big]\Big\}$$

$$C = 0.5772$$

Thus

$$A_{11} = \left[\frac{2}{a} - \frac{1+\cos 2kl}{l} - 2k\,\mathrm{Si}\,2kl\right]$$
$$+ j\left[-2k + \frac{\sin 2kl}{l} - 2k\,\mathrm{Ci}\,2kl + 2kC + 2k\ln 2kl\right]$$

$$A_{12} = \int_0^l \int_0^l \cos k(x_1 - x_2)\left(\frac{\partial^2}{\partial x_1 \partial x_2} - k^2\right) e(r_{12})\, dx_1\, dx_2$$

$$r_{12} = \sqrt{a^2 + x_1^2 + x_2^2 - 2x_1 x_2 \cos 2\alpha}$$

Integrating by parts as before

$$A_{12} = e(r_{ll}) + e(r_{00}) - 2\cos kl\, e(r_{l0}) - 2k\int_0^l \sin kx_1\, e(r_{10})\, dx_1$$
$$- 2k\int_0^l \sin k(l - x_1)\, e(r_{1l})\, dx_1$$

$$\left\{\begin{aligned} r_{ll} &= 2l\sin\alpha \\ r_{00} &= a \\ r_{l0} &= l \end{aligned}\right.$$

$$\int_0^l \sin kx_1\, e(r_{10})\, dx_1 = \frac{1}{2}\int_0^l \frac{\sin 2kx_1}{x_1}\, dx_1 - j\frac{1}{2}\int_0^l \frac{1-\cos 2kx_1}{x_1}\, dx_1$$
$$= \frac{1}{2}\left\{\mathrm{Si}\,2kl - j\left[C + \ln 2kl - \mathrm{Ci}\,2kl\right]\right\}$$

$$r_{1l} = \sqrt{a^2 + x_1^2 + l^2 - 2x_1 l\cos 2\alpha} = \sqrt{a^2 + (x_1 - l\cos 2\alpha)^2 + l^2\sin^2 2\alpha}$$

$$\int_0^l \sin k(l - x_1)\, e(r_{1l})\, dx_1 = \int_0^l \frac{\sin k(l-x_1)\cos kr_{1l}}{r_{1l}}\, dx_1 - j\int_0^l \frac{\sin k(l-x_1)\sin kr_{1l}}{r_{1l}}\, dx_1$$

Let $x = x_1 - l\cos 2\alpha$, $r_x = \sqrt{a^2 + x^2 + l^2\sin^2 2\alpha}$.

$$\int_0^l \sin k(l - x_1)\, e(r_{1l})\, dx_1$$
$$= \int_{-l\cos 2\alpha}^{l - l\cos 2\alpha} \frac{\sin k(l - l\cos 2\alpha - x)\cos kr_x}{r_x}\, dx - j\int_{-l\cos 2\alpha}^{l - l\cos 2\alpha} \frac{\sin k(l - l\cos 2\alpha - x)\sin kr_x}{r_x}\, dx$$

$$= \sin(2kl\sin^2\alpha)\int_{x_0}^{x_l}\frac{\cos kx\cos kr_x}{r_x}dx - \cos(2kl\sin^2\alpha)\int_{x_0}^{x_l}\frac{\sin kx\cos kr_x}{r_x}dx$$

$$-j\sin(2kl\sin^2\alpha)\int_{x_0}^{x_l}\frac{\cos kx\sin kr_x}{r_x}dx + j\cos(2kl\sin^2\alpha)\int_{x_0}^{x_l}\frac{\sin kx\sin kr_x}{r_x}dx$$

$$= \frac{1}{2}\sin(2kl\sin^2\alpha)\Big[Cik(r_l+x_l) - Cik(r_0+x_0) + Cik(r_0-x_0) - Cik(r_l-x_l)\Big]$$

$$-\frac{1}{2}\cos(2kl\sin^2\alpha)\Big[Sik(r_l+x_l) - Sik(r_0+x_0) - Sik(r_0-x_0) + Sik(r_l-x_l)\Big]$$

$$-\frac{j}{2}\sin(2kl\sin^2\alpha)\Big[Sik(r_l+x_l) - Sik(r_0+x_0) + Sik(r_0-x_0) - Sik(r_l-x_l)\Big]$$

$$+\frac{j}{2}\cos(2kl\sin^2\alpha)\Big[Cik(r_0-x_0) - Cik(r_l-x_l) - Cik(r_l+x_l) + Cik(r_0+x_0)\Big]$$

$$\begin{cases} r_0 + x_0 = \sqrt{l^2\cos^2 2\alpha + l^2\sin^2 2\alpha} - l\cos 2\alpha = 2l\sin^2\alpha \\ r_l + x_l = \sqrt{l^2(1-\cos 2\alpha)^2 + l^2\sin^2 2\alpha} + 2l\sin^2\alpha = 2l(\sin\alpha + \sin^2\alpha) \\ r_0 - x_0 = 2l\cos^2\alpha \\ r_l - x_l = 2l(\sin\alpha - \sin^2\alpha) \end{cases}$$

Collecting terms–

$$A_{12} = \Big\{ \frac{\cos(2kl\sin\alpha)}{2l\sin\alpha} + \frac{1}{a} - 2\frac{\cos^2 kl}{l} - kSi2kl$$

$$-k\sin(2kl\sin^2\alpha)\Big[Ci\langle 2kl\sin\alpha(1+\sin\alpha)\rangle - Ci(2kl\sin^2\alpha)$$

$$+Ci(2kl\cos^2\alpha) - Ci\langle 2kl\sin\alpha(1-\sin\alpha)\rangle\Big]$$

$$+k\cos(2kl\sin^2\alpha)\Big[Si\langle 2kl\sin\alpha(1+\sin\alpha)\rangle - Si(2kl\sin^2\alpha)$$

$$-Si(2kl\cos^2\alpha) + Si\langle 2kl\sin\alpha(1-\sin\alpha)\rangle\Big]\Big\}$$

$$+j\Big\{-\frac{\sin(2kl\sin\alpha)}{2l\sin\alpha} - k + 2\frac{\sin kl\cos kl}{l} + kC$$

$$+k\ln 2kl - kCi\,2kl$$

$$+ k\sin(2kl\sin^2\alpha)\Big[Si\langle 2kl\sin\alpha(1+\sin\alpha)\rangle - Si(2kl\sin^2\alpha)$$
$$+ Si(2kl\cos^2\alpha) - Si\langle 2kl\sin\alpha(1-\sin\alpha)\rangle\Big]$$
$$- k\cos(2kl\sin^2\alpha)\Big[Ci(2kl\cos^2\alpha) - Ci\langle 2kl\sin\alpha(1-\sin\alpha)\rangle$$
$$+ Ci(2kl\sin^2\alpha) - Ci\langle 2kl\sin\alpha(1+\sin\alpha)\rangle\Big]\Big\}$$

<u>Note</u>: When $\alpha = 0$, reconsideration of the limits of integration readily shows that A_{12} reduces to A_{11} as it should. This serves as a partial check on these integrations.

$$A_{13} = \int_0^l \int_0^l \cos k(x_1 - x_3 - l)\left(\frac{\partial^2}{\partial x_1 \partial x_3} - k^2\right) e(r_{13})\, dx_1\, dx_3$$

$$r_{13} = \sqrt{a^2 + (x_3 - x_1 + l\cos 2\alpha)^2 + (l\sin 2\alpha)^2}$$

$$\int_0^l \int_0^l \cos k(x_1 - x_3 - l)\frac{\partial^2}{\partial x_1 \partial x_3} e(r_{13})\, dx_1\, dx_3$$

$$= \int_0^l \left[\cos k(x_1 - x_3 - l)\frac{\partial}{\partial x_3} e(r_{13})\right]_0^l dx_3 + k\int_0^l\int_0^l \sin k(x_1 - x_3 - l)\frac{\partial}{\partial x_3} e(r_{13})\, dx_1\, dx_3$$

$$= \int_0^l \cos kx_3 \frac{\partial}{\partial x_3} e(r_{l3})\, dx_3 - \int_0^l \cos k(x_3 + l)\frac{\partial}{\partial x_3} e(r_{03})\, dx_3$$
$$+ k\int_0^l \left[\sin k(x_1 - x_3 - l)\, e(r_{13})\right]_0^l dx_1 + k^2\int_0^l\int_0^l \cos k(x_1 - x_3 - l)\, e(r_{13})\, dx_1\, dx_3$$

$$= \cos kx_3\, e(r_{l3})\Big|_0^l + k\int_0^l \sin kx_3\, e(r_{l3})\, dx_3 - \cos k(x_3 + l)\, e(r_{03})\Big|_0^l$$
$$- k\int_0^l \sin k(x_3 + l)\, e(r_{03})\, dx_3 + k\int_0^l \sin k(x_1 - 2l)\, e(r_{1l})\, dx_1$$
$$- k\int_0^l \sin k(x_1 - l)\, e(r_{10})\, dx_1 + k^2\int_0^l\int_0^l \cos k(x_1 - x_3 - l)\, e(r_{13})\, dx_1\, dx_3$$

$$A_{13} = \cos kl\, e(r_{ll}) - e(r_{l0}) - \cos 2kl\, e(r_{0l}) + \cos kl\, e(r_{00})$$
$$- 2k\int_0^l \sin k(x_3+l)\, e(r_{03})\,dx_3 - 2k\int_0^l \sin k(x_1-l)\, e(r_{10})\,dx_1$$

$$\begin{cases} r_{ll} = l \\ r_{l0} = l\sqrt{(1-\cos 2\alpha)^2 + \sin^2 2\alpha} = 2l\sin\alpha \\ r_{0l} = 2l\cos\alpha \\ r_{00} = l \end{cases}$$

$$2\int_0^l \sin k(x_3+l)\, e(r_{03})\,dx_3$$

$r_{03} = \sqrt{(x_3 + l\cos 2\alpha)^2 + (l\sin 2\alpha)^2}$

Let $x = x_3 + l\cos 2\alpha$

$r_x = \sqrt{x^2 + (l\sin 2\alpha)^2}$

$$= 2\int_{l\cos 2\alpha}^{l+l\cos 2\alpha} \frac{\sin k(x+l-l\cos 2\alpha)\cos kr_x}{r_x}\,dx - j2\int_{x_0}^{x_l} \frac{\sin k(x+l-l\cos 2\alpha)\sin kr_x}{r_x}\,dx$$

$$= 2\left\{\sin(2kl\sin^2\alpha)\int_{x_0}^{x_l}\frac{\cos kx\cos kr_x}{r_x}\,dx + \cos(2kl\sin^2\alpha)\int_{x_0}^{x_l}\frac{\sin kx\cos kr_x}{r_x}\,dx\right\}$$
$$-j2\left\{\sin(2kl\sin^2\alpha)\int_{x_0}^{x_l}\frac{\cos kx\sin kr_x}{r_x}\,dx + \cos(2kl\sin^2\alpha)\int_{x_0}^{x_l}\frac{\sin kx\sin kr_x}{r_x}\,dx\right\}$$

$$= \sin(2kl\sin^2\alpha)\left[Cik(r_l+x_l) - Cik(r_0+x_0) + Cik(r_0-x_0) - Cik(r_l-x_l)\right]$$
$$+ \cos(2kl\sin^2\alpha)\left[Sik(r_l+x_l) - Sik(r_0+x_0) - Sik(r_0-x_0) + Sik(r_l-x_l)\right]$$
$$- j\sin(2kl\sin^2\alpha)\left[Sik(r_l+x_l) - Sik(r_0+x_0) + Sik(r_0-x_0) - Sik(r_l-x_l)\right]$$
$$- j\cos(2kl\sin^2\alpha)\left[Cik(r_0-x_0) - Cik(r_l-x_l) - Cik(r_l+x_l) + Cik(r_0+x_0)\right]$$

$$\begin{cases} r_0 + x_0 = 2l\cos^2\alpha \\ r_l + x_l = 2l\cos\alpha(1+\cos\alpha) \\ r_0 - x_0 = 2l\sin^2\alpha \\ r_l - x_l = 2l\cos\alpha(1-\cos\alpha) \end{cases}$$

$$2\int_0^l \sin k(x_1-l)\, e(r_{10})\,dx_1 \qquad \left\{ \begin{array}{l} r_{10}=\sqrt{(x_1-l\cos 2\alpha)^2+(l\sin 2\alpha)^2} \\ \text{Let} \quad x = x_1 - l\cos 2\alpha \\ \qquad r_x = \sqrt{x^2+(l\sin 2\alpha)^2} \end{array} \right.$$

$$= 2\int_{-l\cos 2\alpha}^{l-l\cos 2\alpha} \frac{\sin k(x-2l\sin^2\alpha)\cos kr_x}{r_x}\,dx - j2\int_{x_0}^{x_l} \frac{\sin k(x-2l\sin^2\alpha)\sin kr_x}{r_x}\,dx$$

= the negative of an integral evaluated for A_{12} above, i.e.,

$$= -\int_0^l \sin k(l-x_1)\, e(r_{12})\,dx_1$$

Collecting results –

$$A_{13} = \Big\{ \frac{1+\cos 2kl}{l} - \frac{\cos(2kl\sin\alpha)}{2l\sin\alpha} - \frac{\cos 2kl\cos(2kl\cos\alpha)}{2l\cos\alpha}$$

$$-k\sin(2kl\sin^2\alpha)\Big[Ci\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - 2Ci(2kl\cos^2\alpha) + 2Ci(2kl\sin^2\alpha)$$

$$-Ci\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - Ci\langle 2kl\sin\alpha(1+\sin\alpha)\rangle + Ci\langle 2kl\sin\alpha(1-\sin\alpha)\rangle\Big]$$

$$-k\cos(2kl\sin^2\alpha)\Big[Si\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - 2Si(2kl\cos^2\alpha) - 2Si(2kl\sin^2\alpha)$$

$$+Si\langle 2kl\cos\alpha(1-\cos\alpha)\rangle + Si\langle 2kl\sin\alpha(1+\sin\alpha)\rangle + Si\langle 2kl\sin\alpha(1-\sin\alpha)\rangle\Big]\Big\}$$

$$+j\Big\{ -\frac{\sin 2kl}{l} + \frac{\sin(2kl\sin\alpha)}{2l\sin\alpha} + \frac{\cos 2kl\sin(2kl\cos\alpha)}{2l\cos\alpha}$$

$$+k\sin(2kl\sin^2\alpha)\Big[Si\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - 2Si(2kl\cos^2\alpha) + 2Si(2kl\sin^2\alpha)$$

$$-Si\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - Si\langle 2kl\sin\alpha(1+\sin\alpha)\rangle + Si\langle 2kl\sin\alpha(1-\sin\alpha)\rangle\Big]$$

$$+k\cos(2kl\sin^2\alpha)\Big[2Ci(2kl\sin^2\alpha) - Ci\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - Ci\langle 2kl\cos\alpha(1+\cos\alpha)\rangle$$

$$+2Ci(2kl\cos^2\alpha) - Ci\langle 2kl\sin\alpha(1-\sin\alpha)\rangle - Ci\langle 2kl\sin\alpha(1+\sin\alpha)\rangle\Big]\Big\}$$

$$A_{14} = \int_0^l \int_0^l \cos k(x_1 - x_4 - l)\left(\frac{\partial^2}{\partial x_1 \partial x_4} - k^2\right) e(r_{14})\, dx_1 dx_4$$

$$r_{14} = \sqrt{a^2 + (l-x_1)^2 + x_4^2 + 2(l-x_1)x_4\cos 2\alpha}$$

Integrating by parts as in A_{13} —

$$A_{14} = \cos kx_4\, e(r_{l4})\Big|_0^l + k\int_0^l \sin kx_4\, e(r_{l4})\, dx_4 - \cos k(x_4+l)\, e(r_{04})\Big|_0^l$$

$$-k\int_0^l \sin k(x_4+l)\, e(r_{04})\, dx_4 + k\int_0^l \sin k(x_1 - 2l)\, e(r_{1l})\, dx_1 - k\int_0^l \sin k(x_1 - l)\, e(r_{10})\, dx_1$$

$$\begin{cases} r_{ll} = l \\ r_{l0} = a \end{cases} \qquad \begin{matrix} r_{0l} = 2l\cos\alpha \\ r_{00} = l \end{matrix}$$

$$A_{14} = \cos kl\, e(r_{ll}) - e(r_{l0}) - \cos 2kl\, e(r_{0l}) + \cos kl\, e(r_{00})$$

$$+2k\int_0^l \sin kx_4\, e(r_{l4})\, dx_4 - 2k\int_0^l \sin k(x_4+l)\, e(r_{04})\, dx_4$$

$$2\int_0^l \sin kx_4\, e(r_{l4})\, dx_4 \qquad \Big\}\; r_{l4} = x_4$$

(as in A_{12})

$$= \mathrm{Si}\, 2kl - j\left[C + \ln 2kl - \mathrm{Ci}\, 2kl\right]$$

$$2\int_0^l \sin k(x_4+l)\, e(r_{04})\, dx_4 \qquad \begin{cases} r_{04} = \sqrt{l^2 + x_4^2 + 2lx_4\cos 2\alpha} \\ \quad = \sqrt{(x_4 + l\cos 2\alpha)^2 + (l\sin 2\alpha)^2} \\ \text{Let} \quad x = x_4 + l\cos 2\alpha \\ \quad r_x = \sqrt{x^2 + (l\sin 2\alpha)^2} \end{cases}$$

(as in A_{13})

$$= \sin(2kl\sin^2\alpha)\Big[\mathrm{Ci}\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - \mathrm{Ci}(2kl\cos^2\alpha) + \mathrm{Ci}(2kl\sin^2\alpha)$$

$$- \mathrm{Ci}\langle 2kl\cos\alpha(1-\cos\alpha)\rangle\Big]$$

$$+\cos(2kl\sin^2\alpha)\Big[\mathrm{Si}\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - \mathrm{Si}(2kl\cos^2\alpha) - \mathrm{Si}(2kl\sin^2\alpha)$$

$$+ \mathrm{Si}\langle 2kl\cos\alpha(1-\cos\alpha)\rangle\Big]$$

$$-j\sin(2kl\sin^2\alpha)\Big[Si\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - Si(2kl\cos^2\alpha) + Si(2kl\sin^2\alpha)$$
$$- Si\langle 2kl\cdot\cos\alpha(1-\cos\alpha)\rangle\Big]$$
$$-j\cos(2kl\sin^2\alpha)\Big[Ci(2kl\sin^2\alpha) - Ci\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - Ci\langle 2kl\cos\alpha(1+\cos\alpha)\rangle$$
$$+ Ci(2kl\cos^2\alpha)\Big]$$

Collecting results—

$$A_{14} = \Big\{\frac{1+\cos 2kl}{l} - \frac{1}{a} - \frac{\cos 2kl\cos(2kl\cos\alpha)}{2l\cos\alpha} + kSi\,2kl$$
$$-k\sin(2kl\sin^2\alpha)\Big[Ci\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - Ci(2kl\cos^2\alpha)$$
$$+ Ci(2kl\sin^2\alpha) - Ci\langle 2kl\cos\alpha(1-\cos\alpha)\rangle\Big]$$
$$-k\cos(2kl\sin^2\alpha)\Big[Si\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - Si(2kl\cos^2\alpha)$$
$$- Si(2kl\sin^2\alpha) + Si\langle 2kl\cos\alpha(1-\cos\alpha)\rangle\Big]\Big\}$$
$$+j\Big\{-\frac{\sin 2kl}{l} + k + \frac{\cos 2kl\sin(2kl\cos\alpha)}{2l\cos\alpha} - kC - k\ln 2kl + kCi\,2kl$$
$$+k\sin(2kl\sin^2\alpha)\Big[Si\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - Si(2kl\cos^2\alpha)$$
$$+ Si(2kl\sin^2\alpha) - Si\langle 2kl\cos\alpha(1-\cos\alpha)\rangle\Big]$$
$$+k\cos(2kl\sin^2\alpha)\Big[Ci(2kl\sin^2\alpha) - Ci\langle 2kl\cos\alpha(1-\cos\alpha)\rangle$$
$$- Ci\langle 2kl\cos\alpha(1+\cos\alpha)\rangle + Ci(2kl\cos^2\alpha)\Big]\Big\}$$

Adding up the above integrations with appropriate signs gives

$$A_{11} - A_{12} - A_{13} + A_{14} \equiv 0,$$

hence the radiation impedance may be computed simply from the following expression,

$$Z_k = j\,240\,k\sin^2\alpha(B_{12} - B_{14}),$$

which corresponds to Equation (22) of Chaney[2].

B_{12} and B_{14} are evaluated below by the method described by Murray[3], and for comparison Murray's notation has been essentially retained. He has furthermore shown that, unless the two linear integration paths lie in the same plane, the impedance integrals will lead to untabulated functions. Therefore, B_{12} and B_{14} have been evaluated by taking all paths in the same plane.

$$B_{12}=\int_0^l\int_0^l \cos k(x_1-x_2)\, e(r_{12})\, dx_1\, dx_2 \quad \left\{ r_{12}=\sqrt{x_1^2+x_2^2-2x_1x_2\cos 2\alpha}\right.$$

$$=\frac{1}{2}\underbrace{\int_0^l\int_0^l \frac{e^{jk(x_1-x_2)}\,e^{-jkr_{12}}}{r_{12}}\,dx_1\,dx_2}_{\stackrel{D}{=}G_1}+\frac{1}{2}\underbrace{\int_0^l\int_0^l \frac{e^{-jk(x_1-x_2)}\,e^{-jkr_{12}}}{r_{12}}\,dx_1\,dx_2}_{\stackrel{D}{=}G_1'}$$

$$=\frac{1}{2}(G_1+G_1')$$

Interchanging x_1 and x_2 gives $G_1=G_1'$, hence

$$B_{12}=G_1$$

$$B_{14}=\int_0^l\int_0^l \cos k(x_1-x_4-l)\, e(r_{14})\, dx_1\, dx_4 \quad \left\{ r_{14}=\sqrt{(l-x_1)^2+x_4^2+2(l-x_1)x_4\cos 2\alpha}\right.$$

Letting $x_1'=l-x_1$, gives

$$B_{14}=\int_0^l\int_0^l \cos k(x_1+x_4)\, e(r_{14})\, dx_1\, dx_4 \quad \left\{ r_{14}=\sqrt{x_1^2+x_4^2+2x_1x_4\cos 2\alpha}\right.$$

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3. F.H. Murray, "Mutual impedance of two skew antenna wires", Proc. I.R.E. ...

$$B_{14} = \frac{1}{2}\underbrace{\int_0^l\int_0^l \frac{e^{jk(x_1+x_4)}\,e^{-jkr_{14}}}{r_{14}}\,dx_1\,dx_4}_{\overset{D}{=}G_2} + \frac{1}{2}\underbrace{\int_0^l\int_0^l \frac{e^{-jk(x_1+x_4)}\,e^{-jkr_{14}}}{r_{14}}\,dx_1\,dx_4}_{\overset{D}{=}G_3}$$

$$= \frac{1}{2}(G_2+G_3)$$

Now the radiation impedance may be expressed

$$Z_r = j\,120k\,\sin^2\alpha\,(2G_1 - G_2 - G_3)$$

Evaluation of G_1:

$$G_1 = \int_0^l\int_0^l \frac{e^{jk(x_1-x_2-r_{12})}}{r_{12}}\,dx_1\,dx_2$$

$$\left.\begin{aligned} r_{12} &= \sqrt{x_1^2+x_2^2-2x_1x_2\cos 2\alpha} \\ &= \sqrt{d_1^2+(x_2-cx_1)^2} \\ &= \sqrt{d_2^2+(x_1-cx_2)^2} \end{aligned}\right\} \begin{aligned} &\text{where} \\ &d_1 \overset{D}{=} \sqrt{x_1^2(1-c^2)} = x_1 s \\ &d_2 \overset{D}{=} \sqrt{x_2^2(1-c^2)} = x_2 s \\ &\text{and} \quad \begin{cases} c \overset{D}{=} \cos 2\alpha \\ s \overset{D}{=} \sin 2\alpha \end{cases} \end{aligned}$$

Let
$$\begin{cases} d_1 t = r_{12} + (x_2 - cx_1) \\ d_1/t = r_{12} - (x_2 - cx_1) \end{cases}$$

Then
$$r_{12} = \frac{d_1}{2}\left(t+\frac{1}{t}\right)$$
$$x_2 - cx_1 = \frac{d_1}{2}\left(t-\frac{1}{t}\right) \qquad \left.\frac{\partial x_2}{\partial t}\right|_{x_1} = \frac{r_{12}}{t}$$

$$G_1 = \int_0^l e^{hx_1}\,dx_1 \int_0^l \frac{e^{-h(x_1+r_{12})}}{r_{12}}\,dx_2 \quad \Big\} \; h \overset{D}{=} jk$$

$$= \int_0^l e^{h(1-c)x_1} dx_1 \int_{t_0}^{t_l} \frac{e^{-hd_1 t}}{t} dt = \int_0^l e^{h(1-c)x_1} dx_1 \int_{kx_1(1-c)}^{k[r_{1l}+(l-cx_1)]} \frac{e^{-ju}}{u} du$$

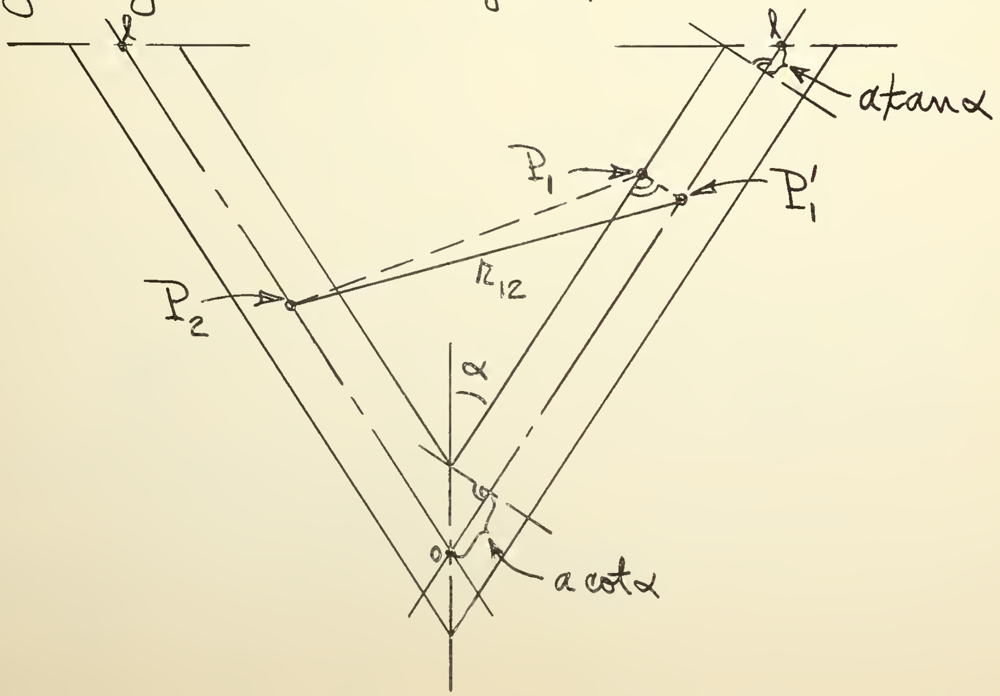
$$= \frac{1}{h(1-c)} \left\{ \underbrace{\left[e^{h(1-c)x_1} \int_{2kx_1\sin^2\alpha}^{k[r_{1l}+(l-cx_1)]} \frac{e^{-ju}}{u} du \right]_0^l}_{\stackrel{D}{=} F(l)-F(0)} - \underbrace{\int_0^l e^{h(1-c)x_1} \left[\frac{e^{-hd_1 t}}{d_1 t} \frac{\partial(d_1 t)}{\partial x_1} \right]_{d_1 t_0}^{d_1 t_l} dx_1}_{\stackrel{D}{=} J_l - J_0} \right\}$$

$$= \frac{1}{h(1-c)} \left\{ F(l) - F(0) - [J_l - J_0] \right\}$$

$$\left[r_{1l} + (l - cx_1) \right]_l = l\sqrt{2(1-c)} + l(1-c) = 2l\sin\alpha(1+\sin\alpha)$$

$$F(l) = e^{j2kl\sin^2\alpha} \Big[Ci\langle 2kl\sin\alpha(1+\sin\alpha)\rangle - Ci(2kl\sin^2\alpha)$$
$$-j\,Si\langle 2kl\sin\alpha(1+\sin\alpha)\rangle + j\,Si(2kl\sin^2\alpha) \Big]$$

Selecting integration limits for x_1:



$$2kx_1\sin^2\alpha\Big]_{a\cot\alpha} = ka\sin 2\alpha$$

$$F(0) = Ci(2kl) - C - \underset{\sim\sim\sim}{\ln(ka\sin 2\alpha)} - jSi(2kl)$$

$$J_l = \int_0^l e^{h(1-c)x_1}\left[\frac{e^{-hd_1t_l}}{d_1t_l}\,\frac{\partial(d_1t_l)}{\partial x_1}\right]dx_1$$

$$\frac{1}{d_1t_l}\,\frac{\partial(d_1t_l)}{\partial x_1} = \frac{1}{r_{1l}+(l-cx_1)}\left(\frac{\partial r_{1l}}{\partial x_1} - c\right)$$

Let

$$\begin{cases} d_l u = r_{1l} - (x_1 - cl) \\ d_l/u = r_{1l} + (x_1 - cl) \end{cases}$$

Then

$$r_{1l} = \frac{d_l}{2}\left(u + \frac{1}{u}\right) \qquad \frac{\partial x_1}{\partial u} = -\frac{r_{1l}}{u}$$

$$x_1 - cl = \frac{d_l}{2}\left(u - \frac{1}{u}\right)$$

$$\frac{\partial r_{1l}}{\partial x_1} - c = \frac{x_1 - cl - cr_{1l}}{r_{1l}}$$

$$x_1 - cl - cr_{1l} = -\frac{d_l}{2}\left[u - \frac{1}{u} + c\left(u + \frac{1}{u}\right)\right] = -\frac{d_l}{2}\left[u(1+c) - \frac{1}{u}(1-c)\right]$$

$$= -\frac{d_l(1+c)}{2u}\left[u^2 - \gamma^2\right] \qquad \gamma^2 \overset{D}{=} \frac{1-c}{1+c}$$

$$\gamma = \tan\alpha$$

$$\frac{dx_1}{r_{1l}} = -\frac{du}{u}$$

$r_{1l} + (l - cx_1)$:

$$cx_1 - (1-s^2)l = -\frac{cd_l}{2}\left(u - \frac{1}{u}\right)$$

$$r_{1l} + l - cx_1 = \frac{d_l}{2}\left[u + \frac{1}{u} + \frac{2s^2l}{d_l} + c\left(u - \frac{1}{u}\right)\right] = \frac{d_l}{2u}(u+\gamma)^2$$

$$J_l = e^{-h(1-c)l}\int_{u_0}^{u_l} \frac{e^{-hd_lu}(u-\tan\alpha)}{u(u+\tan\alpha)}\,du$$

$$\begin{cases} d_l u_l = r_{ll} - l(1-c) = 2l\sin\alpha(1-\sin\alpha) \\ d_l u_0 = l(1+c) = 2l\cos^2\alpha \end{cases}$$

$$J_\ell = e^{-j2k\ell\sin^2\alpha}\left\{-\int_{u_0}^{u_\ell}\frac{e^{-hd_\ell u}}{u}\,du + 2\int_{u_0}^{u_\ell}\frac{e^{-hd_\ell u}}{u+\tan\alpha}\,du\right\}$$

$$= e^{-j2k\ell\sin^2\alpha}\left\{-\int_{2k\ell\cos^2\alpha}^{2k\ell\sin\alpha(1-\sin\alpha)}\frac{e^{-jw}}{w}\,dw \quad + 2e^{j2k\ell\sin^2\alpha}\int_{2k\ell}^{2k\ell\sin\alpha}\frac{e^{-jw}}{w}\,dw\right\}$$

$$\left\{\begin{array}{l} d_\ell u_\ell + d_\ell\tan\alpha = 2\ell\sin\alpha(1-\sin\alpha) + 2\ell\sin^2\alpha = 2\ell\sin\alpha \\ d_\ell u_0 + d_\ell\tan\alpha = 2\ell \end{array}\right.$$

$$J_\ell = -e^{-j2k\ell\sin^2\alpha}\Big[Ci\langle 2k\ell\sin\alpha(1-\sin\alpha)\rangle - Ci(2k\ell\cos^2\alpha)$$
$$-jSi\langle 2k\ell\sin\alpha(1-\sin\alpha)\rangle + jSi(2k\ell\cos^2\alpha)\Big]$$
$$+2\Big[Ci(2k\ell\sin\alpha) - Ci(2k\ell) - jSi(2k\ell\sin\alpha) - jSi(2k\ell)\Big]$$

$$d_1 t_0 = x_{10} - cx_1 = x_1(1-c)$$

$$J_0 = \int_0^\ell e^{h(1-c)x_1}\left[\frac{e^{-hd_1t_0}}{d_1t_0}\,\frac{\partial(d_1t_0)}{\partial x_1}\right]dx_1 = \int_0^\ell e^{h(1-c)x_1}\left[\frac{e^{-h(1-c)x_1}}{(1-c)x_1}(1-c)\right]dx_1$$

$$= \int_{a\cot\alpha}^{\ell}\frac{dx_1}{x_1} = \ln\left(\frac{\ell\tan\alpha}{a}\right)$$

Collecting results

$$G_1 = \frac{1}{j2k\sin^2\alpha}\Big\{\ln(2k\ell\sin^2\alpha) + C + Ci(2k\ell) - 2\,Ci(2k\ell\sin\alpha)$$
$$+\cos(2k\ell\sin^2\alpha)\Big[Ci\langle 2k\ell\sin\alpha(1+\sin\alpha)\rangle - Ci(2k\ell\sin^2\alpha)\Big]$$
$$+\sin(2k\ell\sin^2\alpha)\Big[Si\langle 2k\ell\sin\alpha(1+\sin\alpha)\rangle - Si(2k\ell\sin^2\alpha)\Big]$$
$$+\cos(2k\ell\sin^2\alpha)\Big[Ci\langle 2k\ell\sin\alpha(1-\sin\alpha)\rangle - Ci(2k\ell\cos^2\alpha)\Big]$$
$$-\sin(2k\ell\sin^2\alpha)\Big[Si\langle 2k\ell\sin\alpha(1-\sin\alpha)\rangle - Si(2k\ell\cos^2\alpha)\Big]$$

$$+j2\,Si(2kl\sin\alpha) - j\,Si(2kl)$$
$$-j\cos(2kl\sin^2\alpha)\left[Si\langle 2kl\sin\alpha(1+\sin\alpha)\rangle - Si(2kl\sin^2\alpha)\right]$$
$$+j\sin(2kl\sin^2\alpha)\left[Ci\langle 2kl\sin\alpha(1+\sin\alpha)\rangle - Ci(2kl\sin^2\alpha)\right]$$
$$-j\cos(2kl\sin^2\alpha)\left[Si\langle 2kl\sin\alpha(1-\sin\alpha)\rangle - Si(2kl\cos^2\alpha)\right]$$
$$-j\sin(2kl\sin^2\alpha)\left[Ci\langle 2kl\sin\alpha(1-\sin\alpha)\rangle - Ci(2kl\cos^2\alpha)\right]\Bigg\}$$

$$G_2 = \int_0^l\int_0^l \frac{e^{jk(x_1+x_4-r_{14})}}{r_{14}}\,dx_1\,dx_4$$

$$r_{14} = \sqrt{x_1^2+x_4^2+2x_1x_4c} = \sqrt{d_1^2+(x_4+cx_1)^2} = \sqrt{d_4^2+(x_1+cx_4)^2}$$

where
$$d_1 \stackrel{D}{=} x_1 s \qquad d_4 \stackrel{D}{=} x_4 s$$

$$d_1 t = r_{14} - (x_4+cx_1) \qquad r_{14} = \frac{d_1}{2}\left(t+\frac{1}{t}\right)$$
$$d_1/t = r_{14} + (x_4+cx_1) \qquad x_4+cx_1 = -\frac{d_1}{2}\left(t-\frac{1}{t}\right)$$
$$\frac{\partial x_4}{\partial t} = -\frac{r_{14}}{t}$$

$$G_2 = \int_0^l e^{hx_1}dx_1\int_0^l \frac{e^{h(x_4-r_{14})}}{r_{14}}\,dx_4 = -\int_0^l e^{h(1-c)x_1}dx_1\int_{t_0}^{t_l}\frac{e^{-hd_1t}}{t}\,dt$$
$$= -\int_0^l e^{h(1-c)x_1}dx_1\int_{2kx_1\sin^2\alpha}^{k[r_{1l}-(l-cx_1)]}\frac{e^{-ju}}{u}\,du$$
$$= -\frac{1}{h(1-c)}\left\{\underbrace{\left[e^{h(1-c)x_1}\int_{kd_1t_0}^{kd_1t_l}\frac{e^{-ju}}{u}\,du\right]_0^l}_{F(l)-F(0)} - \underbrace{\int_0^l e^{h(1-c)x_1}\left[\frac{e^{-hd_1t}}{d_1t}\,\frac{\partial(d_1t)}{\partial x_1}\right]_{d_1t_0}^{d_1t_l}dx_1}_{J_l - J_0}\right\}$$

$$F(l) = e^{jk(1-c)l}\Big\{Ci\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - Ci(2kl\sin^2\alpha)$$
$$-jSi\langle 2kl\cos\alpha(1-\cos\alpha)\rangle + jSi(2kl\sin^2\alpha)\Big\}$$

$$r_{ll} = 2l\cos\alpha$$

$$r_{ll} - l(1+c) = 2l\cos\alpha(1-\cos\alpha)$$

$$\left[r_{1l} - (l+cx_1)\right]_{a\tan\alpha} = \sqrt{l^2 + 2al\tan\alpha\cos 2\alpha + a^2\tan^2\alpha} - (l + a\tan\alpha\cos 2\alpha)$$
$$= \frac{2a^2}{l}\sin^4\alpha$$

$$\left[2x_1\sin^2\alpha\right]_{a\tan\alpha} = 2a\frac{\sin^3\alpha}{\cos\alpha}$$

$$F(0) = Ci\left(\frac{2ka^2}{l}\sin^4\alpha\right) - Ci\left(2ka\frac{\sin^3\alpha}{\cos\alpha}\right) = \ln\left(\frac{a}{l}\sin\alpha\cos\alpha\right)$$

$$J_l = \int_0^l e^{h(1-c)x_1}\left[\frac{e^{-hd_1t_l}}{d_1t_l}\frac{\partial(d_1t_l)}{\partial x_1}\right]dx_1$$

$$\frac{1}{d_1t_l}\frac{\partial(d_1t_l)}{\partial x_1} = \frac{1}{r_{1l}-(l+cx_1)}\left(\frac{\partial r_{1l}}{\partial x_1} - c\right)$$

$$d_l u = r_{1l} - (x_1 + cl) \qquad r_{1l} = \frac{d_l}{2}\left(u + \frac{1}{u}\right)$$

$$d_l/u = r_{1l} + (x_1 + cl) \qquad x_1 + cl = -\frac{d_l}{2}\left(u - \frac{1}{u}\right)$$

$$\frac{\partial x_1}{\partial u} = -\frac{r_{1l}}{u}$$

$$\frac{\partial r_{1l}}{\partial x_1} - c = \frac{x_1 - cl - cr_{1l}}{r_{1l}}$$

$$x_1 - cl - cr_{1l} = -\frac{d_l}{2}\left[u - \frac{1}{u} + c\left(u + \frac{1}{u}\right)\right]$$

$$= -\frac{d_l(1+c)}{2u}(u^2 - \gamma^2) \qquad \gamma^2 = \frac{1-c}{1+c}$$

$$\frac{dx_1}{r_{1l}} = -\frac{du}{u} \qquad \gamma = \tan\alpha$$

$r_{1\ell} - \ell - cx_1$: $\qquad cx_1 + (1-s^2)\ell = -\frac{d_\ell c}{2}\left(u - \frac{1}{u}\right)$

$$r_{1\ell} - \ell - cx_1 = \frac{d_\ell}{2}\left[u + \frac{1}{u} - \frac{2\ell s^2}{d_\ell} + c\left(u - \frac{1}{u}\right)\right]$$

$$= \frac{d_\ell(1+c)}{2u}\left(u - \gamma\right)^2$$

$$J_\ell = e^{jk(1-c)\ell}\int_{u_0}^{u_\ell} \frac{e^{-hd_\ell u}\,(u+\tan\alpha)}{u(u-\tan\alpha)}\,du$$

$$d_\ell u_\ell = r_{\ell\ell} - \ell(1+c) = 2\ell\cos\alpha(1-\cos\alpha)$$

$$d_\ell u_0 = r_{0\ell} - c\ell = 2\ell\sin^2\alpha$$

$$J_\ell = e^{jk(1-c)\ell}\left\{-\int_{u_0}^{u_\ell}\frac{e^{-hd_\ell u}}{u}\,du + 2\int_{u_0}^{u_\ell}\frac{e^{-hd_\ell u}}{u-\tan\alpha}\,du\right\}$$

$$= e^{j2k\ell\sin^2\alpha}\left\{-\int_{kd_\ell u_0}^{kd_\ell u_\ell}\frac{e^{-jw}}{w}\,dw + 2e^{-hd_\ell\tan\alpha}\int_{kd_\ell(\tan\alpha-u_0)}^{kd_\ell(\tan\alpha-u_\ell)}\frac{e^{+jw}}{w}\,dw\right\}$$

$$d_\ell(\tan\alpha - u_\ell) = d_\ell\tan\alpha - d_\ell u_\ell = 2\ell(1-\cos\alpha)$$

$$d_\ell(\tan\alpha - u) = 2\ell\sin^2\alpha - \sqrt{\ell^2 + 2\ell c x_1 + x_1^2} + (x_1 + c\ell)$$

$$d_\ell(\tan\alpha - u)\Big|_{a\tan\alpha} = \frac{2ka\sin^3\alpha}{\cos\alpha}$$

$$J_\ell = \underbrace{-e^{j2k\ell\sin^2\alpha}\int_{2k\ell\sin^2\alpha}^{2k\ell\cos\alpha(1-\cos\alpha)}\frac{e^{-jw}}{w}\,dw}_{F(\ell)} + 2\int_{\frac{2ka\sin^3\alpha}{\cos\alpha}}^{2k\ell(1-\cos\alpha)}\frac{e^{+jw}}{w}\,dw$$

$$= -F(\ell) + 2\left\{Ci\langle 2k\ell(1-\cos\alpha)\rangle + j\,Si\langle 2k\ell(1-\cos\alpha)\rangle - C - \ln\frac{2ka\sin^3\alpha}{\cos\alpha}\right\}$$

$$d_1 t_0 = r_{10} - cx_1 = x_1(1-c)$$

$$J_0 = \int_0^\ell e^{h(1-c)x_1}\left[\frac{e^{-h(1-c)x_1}}{x_1\cancel{(1-c)}}\cancel{(1-c)}\right]dx_1 = \int_{a\tan\alpha}^{\ell}\frac{dx_1}{x_1}$$

$$= \ln\left(\frac{\ell}{a\tan\alpha}\right)$$

$$G_2 = -\frac{1}{j2k\sin^2\alpha}\left\{F(l) - F(0) - J_l + J_0\right\}$$

$$= -\frac{1}{j2k\sin^2\alpha}\Big\{2\ln(2kl\sin^2\alpha) + 2C - 2Ci\langle 2kl(1-\cos\alpha)\rangle - 2\ln(\cos\alpha)$$

$$+ 2\cos(2kl\sin^2\alpha)\left[Ci\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - Ci(2kl\sin^2\alpha)\right]$$

$$+ 2\sin(2kl\sin^2\alpha)\left[Si\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - Si(2kl\sin^2\alpha)\right]$$

$$-j2\cos(2kl\sin^2\alpha)\left[Si\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - Si(2kl\sin^2\alpha)\right]$$

$$+j2\sin(2kl\sin^2\alpha)\left[Ci\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - Ci(2kl\sin^2\alpha)\right]$$

$$-j2Si\langle 2kl(1-\cos\alpha)\rangle\Big\}$$

$$G_3 = \int_0^l\int_0^l \frac{e^{-jk(x_1+x_4+r_{14})}}{r_{14}}\,dx_1\,dx_4$$

$$d_1 t = r_{14} + (x_4 + cx_1) \qquad r_{14} = \frac{d_1}{2}\left(t + \frac{1}{t}\right)$$

$$d_1/t = r_{14} - (x_4 + cx_1) \qquad x_4 + cx_1 = \frac{d_1}{2}\left(t - \frac{1}{t}\right)$$

$$\frac{\partial x_4}{\partial t} = \frac{r_{14}}{t}$$

$$G_3 = \int_0^l e^{-h(1-c)x_1}dx_1\int_{t_0}^{t_l}\frac{e^{-hd_1t}}{t}\,dt = \int_0^l e^{-h(1-c)x_1}dx_1\int_{kd_1t_0}^{kd_1t_l}\frac{e^{-ju}}{u}\,du$$

$$= -\frac{1}{h(1-c)}\Bigg\{\underbrace{\left[e^{-h(1-c)x_1}\int_{2kx_1\cos^2\alpha}^{k[r_{1l}+(l+cx_1)]}\frac{e^{-ju}}{u}\,du\right]_0^l}_{F(l)-F(0)} - \underbrace{\int_0^l e^{-h(1-c)x_1}\left[\frac{e^{-hd_1t}}{d_1t}\frac{\partial(d_1t)}{\partial x_1}\right]_{d_1t_0}^{d_1t_l}dx_1}_{J_l - J_0}\Bigg\}$$

$$r_{ll} = 2l\cos\alpha$$

$$r_{ll} + l(1+c) = 2l\cos\alpha\,(1+\cos\alpha)$$

$$F(\ell) = e^{-j2k\ell\sin^2\alpha}\Big\{Ci\big\langle 2k\ell\cos\alpha(1+\cos\alpha)\big\rangle - Ci(2k\ell\cos^2\alpha)$$
$$-jSi\big\langle 2k\ell\cos\alpha(1+\cos\alpha)\big\rangle + jSi(2k\ell\cos^2\alpha)\Big\}$$

$$\left[r_{1\ell}+(\ell+cx_1)\right]_0 = 2\ell$$

$$2kx_1\cos^2\alpha\Big|_{a\tan\alpha} = ka\sin 2\alpha$$

$$F(0) = Ci(2k\ell) - C - \ln(ka\sin 2\alpha) - jSi(2k\ell)$$

$$J_\ell = \int_0^\ell e^{-h(1-c)x_1}\left[\frac{e^{-hd_1t_\ell}}{d_1t_\ell}\frac{\partial(d_1t_\ell)}{\partial x_1}\right]dx_1$$

$$\frac{1}{d_1t_\ell}\frac{\partial(d_1t_\ell)}{\partial x_1} = \frac{1}{r_{1\ell}+(\ell+cx_1)}\left(\frac{\partial r_{1\ell}}{\partial x_1}+c\right)$$

$$d_\ell u = r_{1\ell}+(x_1+c\ell) \qquad r_{1\ell} = \frac{d_\ell}{2}\left(u+\frac{1}{u}\right)$$
$$d_\ell/u = r_{1\ell}-(x_1+c\ell) \qquad x_1+c\ell = \frac{d_\ell}{2}\left(u-\frac{1}{u}\right)$$
$$\frac{\partial x_1}{\partial u} = \frac{r_{1\ell}}{u}$$

$$\frac{\partial r_{1\ell}}{\partial x_1}+c = \frac{x_1+c\ell+cr_{1\ell}}{r_{1\ell}}$$

$$x_1+c\ell+cr_{1\ell} = \frac{d_\ell}{2}\left[u-\frac{1}{u}+c\left(u+\frac{1}{u}\right)\right]$$
$$= \frac{d_\ell(1+c)}{2u}(u^2-\gamma^2) \quad \Bigg\}\ \begin{array}{l}\gamma^2 = \frac{1-c}{1+c}\\ \gamma = \tan\alpha\end{array}$$

$$\frac{dx_1}{r_{1\ell}} = \frac{du}{u}$$

$r_{1\ell}+\ell+cx_1$: $\qquad cx_1+(1-s^2)\ell = \dfrac{cd_\ell}{2}\left(u-\dfrac{1}{u}\right)$

$$r_{1\ell}+\ell+cx_1 = \frac{d_\ell}{2}\left[u+\frac{1}{u}+\frac{2s^2\ell}{d_\ell}+c\left(u-\frac{1}{u}\right)\right]$$
$$= \frac{d_\ell}{2u}(u+\gamma)^2$$

$$J_\ell = e^{-h(1-c)\ell}\int_{u_0}^{u_\ell} \frac{e^{-hd_\ell u}(u-\tan\alpha)}{u(u+\tan\alpha)}\,du$$

$$= e^{-j2k\ell\sin^2\alpha}\left\{-\int_{u_0}^{u_\ell}\frac{e^{-hd_\ell u}}{u}\,du + 2\int_{u_0}^{u_\ell}\frac{e^{-hd_\ell u}}{u+\tan\alpha}\,du\right\}$$

$$= e^{-j2k\ell\sin^2\alpha}\left\{-\int_{d_\ell u_0}^{d_\ell u_\ell}\frac{e^{-jw}}{w}\,dw + 2e^{j2k\ell\sin^2\alpha}\int_{d_\ell(u_0+\tan\alpha)}^{d_\ell(u_\ell+\tan\alpha)}\frac{e^{-jw}}{w}\,dw\right\}$$

$$d_\ell u_\ell = r_{\ell\ell} + (1+c)\ell = 2\ell\cos\alpha(1+\cos\alpha)$$

$$d_\ell u_0 = r_{0\ell} + c\ell = 2\ell\cos^2\alpha$$

$$d_\ell u_\ell + d_\ell\tan\alpha = 2\ell(1+\cos\alpha)$$

$$d_\ell u_0 + d_\ell\tan\alpha = 2\ell$$

$$J_\ell = -F(\ell) + 2\Big[Ci\langle 2k\ell(1+\cos\alpha)\rangle - Ci(2k\ell) - jSi\langle 2k\ell(1+\cos\alpha)\rangle + jSi(2k\ell)\Big]$$

$$d_1 t_0 = r_{10} + cx_1 = x_1(1+c)$$

$$J_0 = \int_0^\ell e^{-h(1-c)x_1}\left[\frac{e^{-h(1+c)x_1}}{x_1\cancel{(1+c)}}\cancel{(1+c)}\right]dx_1 = \int_0^\ell \frac{e^{-j2kx_1}}{x_1}\,dx_1$$

$$= \int_{2ka\tan\alpha}^{2k\ell}\frac{e^{-jw}}{w}\,dw$$

$$= Ci(2k\ell) - C - \ln(2ka\tan\alpha) - jSi(2k\ell)$$

$$G_3 = -\frac{1}{j2k\sin^2\alpha}\Big\{2Ci(2k\ell) + 2\ln(\cos\alpha) - 2Ci\langle 2k\ell(1+\cos\alpha)\rangle$$
$$+ 2\cos(2k\ell\sin^2\alpha)\Big[Ci\langle 2k\ell\cos\alpha(1+\cos\alpha)\rangle - Ci(2k\ell\cos^2\alpha)\Big]$$
$$- 2\sin(2k\ell\sin^2\alpha)\Big[Si\langle 2k\ell\cos\alpha(1+\cos\alpha)\rangle - Si(2k\ell\cos^2\alpha)\Big]$$

$$+j\left[2\,\mathrm{Si}\langle 2kl(1+\cos\alpha)\rangle - 2\,\mathrm{Si}(2kl)\right]$$
$$-j2\cos(2kl\sin^2\alpha)\left[\mathrm{Si}\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - \mathrm{Si}(2kl\cos^2\alpha)\right]$$
$$-j2\sin(2kl\sin^2\alpha)\left[\mathrm{Ci}\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - \mathrm{Ci}(2kl\cos^2\alpha)\right]$$

Now from p. 15

$$\frac{Z_R}{120} = jk\sin^2\alpha(2G_1 - G_2 - G_3)$$

Substituting for the G's now gives

$$\begin{aligned}
\frac{Z_R}{120} = {} & 2C + 2\ln(2kl\sin^2\alpha) + 2\,\mathrm{Ci}(2kl) - 2\,\mathrm{Ci}(2kl\sin\alpha) \\
& - \mathrm{Ci}\langle 2kl(1+\cos\alpha)\rangle - \mathrm{Ci}\langle 2kl(1-\cos\alpha)\rangle \\
& + \cos(2kl\sin^2\alpha)\Big[\mathrm{Ci}\langle 2kl\cos\alpha(1-\cos\alpha)\rangle + \mathrm{Ci}\langle 2kl\sin\alpha(1-\sin\alpha)\rangle \\
& \quad + \mathrm{Ci}\langle 2kl\cos\alpha(1+\cos\alpha)\rangle + \mathrm{Ci}\langle 2kl\sin\alpha(1+\sin\alpha)\rangle \\
& \quad - 2\,\mathrm{Ci}(2kl\cos^2\alpha) - 2\,\mathrm{Ci}(2kl\sin^2\alpha)\Big] \\
& + \sin(2kl\sin^2\alpha)\Big[\mathrm{Si}\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - \mathrm{Si}\langle 2kl\sin\alpha(1-\sin\alpha)\rangle \\
& \quad - \mathrm{Si}\langle 2kl\cos\alpha(1+\cos\alpha)\rangle + \mathrm{Si}\langle 2kl\sin\alpha(1-\sin\alpha)\rangle \\
& \quad + 2\,\mathrm{Si}(2kl\cos^2\alpha) - 2\,\mathrm{Si}(2kl\sin^2\alpha)\Big] \\
& + j\Big\{\mathrm{Si}\langle 2kl(1+\cos\alpha)\rangle - \mathrm{Si}\langle 2kl(1-\cos\alpha)\rangle + 2\,\mathrm{Si}(2kl\sin\alpha) \\
& \quad - 2\,\mathrm{Si}(2kl) \\
& + \cos(2kl\sin^2\alpha)\Big[-\mathrm{Si}\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - \mathrm{Si}\langle 2kl\sin\alpha(1-\sin\alpha)\rangle \\
& \quad - \mathrm{Si}\langle 2kl\cos\alpha(1+\cos\alpha)\rangle - \mathrm{Si}\langle 2kl\sin\alpha(1+\sin\alpha)\rangle \\
& \quad + 2\,\mathrm{Si}(2kl\cos^2\alpha) + 2\,\mathrm{Si}(2kl\sin^2\alpha)\Big] \\
& + \sin(2kl\sin^2\alpha)\Big[\mathrm{Ci}\langle 2kl\cos\alpha(1-\cos\alpha)\rangle - \mathrm{Ci}\langle 2kl\sin\alpha(1-\sin\alpha)\rangle \\
& \quad - \mathrm{Ci}\langle 2kl\cos\alpha(1+\cos\alpha)\rangle + \mathrm{Ci}\langle 2kl\sin\alpha(1+\sin\alpha)\rangle \\
& \quad + 2\,\mathrm{Ci}(2kl\cos^2\alpha) - 2\,\mathrm{Ci}(2kl\sin^2\alpha)\Big]\Big\}
\end{aligned}$$

RADIATION IMPEDANCE OF THE TERMINATED VEE IN FREE SPACE

The radiation impedance of the vee may be readily computed now from Chaney's formula for the input impedance of the closed circuit* except that now the open end of the vee is assumed closed through the terminating impedance. If the leg length is l and the vertex angle 2α as before, and the same notation is employed as for the rhombic, the obvious result is obtained

$$Z_R = 2(Z_{11} - Z_{12})$$

$$= \frac{60}{jk}(A_{11} - A_{12} - 2k^2 \sin^2\alpha\, B_{12})$$

or

$$\frac{Z_R}{60} = \frac{1}{jk}(A_{11} - A_{12}) + j\,2k \sin^2\alpha\, G_1$$

Substitution gives

$$\frac{Z_R}{60} = \frac{\sin(2kl\sin\alpha)}{2kl\sin\alpha} - 1 + 2\left[C + \ln(2kl\sin\alpha) - Ci(2kl\sin\alpha)\right]$$

$$+ j\left[2\,Si(2kl\sin\alpha) + \frac{\cos(2kl\sin\alpha)}{2kl\sin\alpha} - \frac{1}{ka}\right]$$

This expression may be simplified by introducing the modified cosine integral

$$Cin\, x = C + \ln x - Ci\, x$$

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* See p.2.

and by noting that the reactive components

$$\frac{\cos(2kl\sin\alpha)}{2kl\sin\alpha} - \frac{1}{ka}$$

represent the capacitance existing between the open ends of the vee. Since this capacitive reactance is shunted by the terminating impedance Z_0, these latter terms may be discarded as being negligible. Thus finally

$$\frac{Z_R}{60} = \frac{\sin(2kl\sin\alpha)}{2kl\sin\alpha} - 1 + 2\,\mathrm{Cin}(2kl\sin\alpha) + j2\,\mathrm{Si}(2kl\sin\alpha)$$